

Zamolodchikov operator-valued relations for $SL(2, R)_k$ WZNW model

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ABSTRACT

An infinite set of operator-valued relations that hold for reducible representations of the $\widehat{sl}(2)_k$ algebra is derived. These relations are analogous to those recently obtained by Zamolodchikov which involve logarithmic fields associated to the Virasoro degenerate representations in Liouville theory. The fusion rules of the $\widehat{sl}(2)_k$ algebra turn out to be a crucial step in the analysis. The possible relevance of these relations for the boundary theory in the AdS_3/CFT_2 correspondence is suggested.

1 Introduction

In [1], Al. Zamolodchikov proved the existence of a set of operator-valued relations in Liouville field theory. There is one such relation for every degenerate Virasoro primary field, which is labelled by a pair of positive integers (m, n) . These relations correspond to a higher order generalization of the Liouville equation of motion.

In this letter, we prove that a similar set of relations holds for Kac-Moody degenerate representations in conformal theories with $sl(\hat{2})_k$ affine symmetry. These operator-valued identities are translated into differential equations satisfied by correlators involving states of reducible representations in the $SL(2, \mathbb{R})_k$ WZNW model. These are differential equations in terms of the (x, \bar{x}) $SL(2, \mathbb{R})$ -isospin variables. Schematically

$$\bar{K}_{m,n} K_{m,n} \Phi'_{j_{m,n}}(x, \bar{x}|z, \bar{z}) = B_{m,n} \Phi_{\tilde{j}_{m,n}}(x, \bar{x}|z, \bar{z}), \quad (1.1)$$

where $\Phi'_{j_{m,n}}$ are the logarithmic fields associated to the degenerate Kac-Moody primaries $\Phi_{j_{m,n}}$. By $\bar{K}_{m,n}$ and $K_{m,n}$ we denote the left and right operators that create the null vector in the Verma module generated by $\Phi_{j_{m,n}}$. Finally, $\Phi_{\tilde{j}_{m,n}}$ is a Kac-Moody primary and $B_{m,n}$ is a c -number we will call *Zamolodchikov coefficient*.

The WZNW model on $SL(2, \mathbb{R})$ represents the worldsheet CFT describing string theory on AdS_3 target space. This model has been extensively studied in this context and in relation to exact string backgrounds. The string spectrum in AdS_3 is constructed in terms of continuous and discrete irreducible representations of $SL(2, \mathbb{R})_k$ (see [2] and references therein); these representations are typically classified by a complex number ¹ j . In particular, unitarity of the theory requires a truncation of the set of discrete representations which is done by imposing the bound

$$\frac{1-k}{2} < j < -\frac{1}{2}. \quad (1.2)$$

The consideration of three and four-point functions leads to even more restrictive constraints on the set of states involved in the correlators.

Here, we focus our attention on reducible representations of $SL(2, \mathbb{R})_k$. These represen-

¹See below for detailed description of the representations.

tations are classified by special values of the index $j = j_{m,n}^\pm$, namely

$$j_{m,n}^+ = \frac{m-1}{2} + \frac{n-1}{2}(k-2), \quad j_{m,n}^- = -\frac{m+1}{2} - \frac{n}{2}(k-2), \quad (1.3)$$

for (m, n) a pair of positive integers.

Certainly, these values of j do not belong to the range (1.2) and therefore do not represent perturbative string states in AdS_3 . However, states belonging to (1.3) were previously considered in this theory; for instance, four-point functions involving the state $j_{1,1}^- = -\frac{k}{2}$ were considered in [2] in order to construct three-point string amplitudes violating the winding number. The corresponding vertex operator $\Phi_{j_{1,1}^-}$ was referred to as the *spectral flow operator*. Also in [3] the highest-weight state of this representation was considered as the conjugate representation of the identity operator $\Phi_{j_{1,1}^+}$.

In [4], the admissible representations of $SL(2, \mathbb{R})_k$ were studied in relation to a certain identity existing between correlators of WZNW theories and minimal models. By analyzing the correlators, Andreev was able to reobtain the fusion rules for these representations originally found by Awata and Yamada in [5].

The states $\Phi_{j_{3,2}^+}$, $\Phi_{j_{2,2}^+}$ and $\Phi_{j_{2,1}^+}$ were considered in [6] as particular examples to discuss the explicit form of the four-point functions in $sl(\hat{2})_k$ and $su(\hat{2})_k$ models. Four-point functions involving generic states of admissible representations were also discussed in detail in references [7]. Furthermore, the representation $j_{2,1}^+$ was employed by Teshner to compute three-point functions in $SL(2, \mathbb{C})/SU(2)$ WZNW model by using the bootstrap approach [8]; the representation $j_{1,2}^+$ was also discussed there. The field $\Phi_{j_{2,1}^+}$ was also studied in [9] in the context of the path integral approach to string theory in AdS_3 .

These representations were also considered in reference [10] in the context of D -branes in AdS_3 string theory, where solutions analogous to the ZZ branes of Liouville theory were discussed (see [11] for related discussions).

Notice that only the values $j_{m,1}^+$ are k -independent and therefore finite in the classical limit, $k \rightarrow \infty$. We will refer to these particular representations as the *classical branch*. In this particular case, it is possible to verify directly that the above relations hold at the classical level.

The paper is organized as follows: In section 2, we discuss the $sl(\hat{2})_k$ affine algebra and its unitary representations. We review the formula for three-point correlation functions and

we list some facts about reducible representations. In section 3, we introduce logarithmic operators in the CFT and provide a preliminary argument that the above operator-valued relations hold. We also show that they are verified at the classical level for the operators in the *classical branch*. Section 4 contains a complete proof and the explicit formulas for the Zamolodchikov coefficients $B_{m,n}^\pm$. We dedicate section 5 to the discussion of the results.

2 The $sl(\hat{2})_k$ affine algebra

2.1 Kac-Moody algebra and unitary representations

The $sl(\hat{2})_k$ affine algebra is defined by the following Lie products

$$[J_n^3, J_m^\pm] = \pm J_{n+m}^\pm, \quad [J_n^3, J_m^3] = -\frac{k}{2}n\delta_{n,-m}, \quad [J_n^+, J_m^-] = 2J_{n+m}^3 - kn\delta_{n,-m} \quad (2.4)$$

By defining the operators

$$L_n = -\frac{1}{k-2} \sum_{n \in \mathbb{Z}} : J_n^+ J_{m-n}^- + J_n^- J_{m-n}^+ + 2J_n^3 J_{m-n}^3 :$$

it is feasible to show that we obtain a representation of the Virasoro algebra with central charge c given by

$$c = 3 + \frac{6}{k-2}$$

Here, we will consider the case $k > 2$.

As usual, we can encode the $sl(\hat{2})_k$ structure in the operator product expansion of local operators $J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n^a$, $a = \{+, -, 3\}$.

Representations of $SL(2, \mathbb{R})$ are classified by an index j and the vectors $|j, m\rangle$ are labelled by an index m . Hermitian unitary representations are listed below:

Highest weight discrete series \mathcal{D}_j^+ : In these infinite dimensional representations $2j \in \mathbb{N}$ and $m \in j - \mathbb{N}$.

Lowest weight discrete series \mathcal{D}_j^- : These are analogous to the highest weight series, being $2j \in \mathbb{N}$ and $m \in \mathbb{N} - j$.

Complementary series \mathcal{E}_j^α : These are defined for $j \in (\frac{1}{2}, 1)$ and $2j > -1 - |2\alpha - 1|$, where $\alpha \in (0, 1]$ and $m \in \alpha + \mathbb{Z}$.

Principal continuous series \mathcal{C}_t^α : These are defined for $j \in -\frac{1}{2} + it$, with $t \in \mathbb{R}$, $\alpha \in (0, 1]$ and $m \in \alpha + \mathbb{Z}$.

Identity representation \mathcal{I} : This is defined for $j = m = 0$.

The spectrum of j can be restricted to $j < -\frac{1}{2}$ by considering the invariance under Weyl reflection $j \leftrightarrow -1 - j$. Moreover, the indices of discrete representations \mathcal{D}_j^\pm have to be bounded from below as $\frac{1-k}{2} < j$ in order to guarantee the non-negative norm condition for the states of the Kac-Moody module.

The representations of the universal covering of $SL(2, \mathbb{R})$ are defined by the relaxation of the condition $2j \in \mathbb{N} \rightarrow \mathbb{R}$ for the discrete series \mathcal{D}_j^\pm .

The Kac-Moody primary states $|j, m\rangle$ are classified in terms of the mentioned representations and satisfy the following properties

$$J_0^3 |j, m\rangle = m |j, m\rangle, \quad J_0^\pm |j, m\rangle = (\mp j - m) |j, m \pm 1\rangle, \quad J_{n>0}^a |j, m\rangle = 0 \quad (2.5)$$

where $a = \{+, -, 3\}$. These states are primary states of the Virasoro algebra, namely

$$L_0 |j, m\rangle = \Delta_j |j, m\rangle, \quad L_{n>0} |j, m\rangle = 0$$

where the L_n are the Fourier modes of the Sugawara stress-tensor $T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$ and $\Delta_j = -\frac{j(j+1)}{k-2}$. Notice that the spectrum (which basically is given by the value of the quadratic Casimir) remains invariant under Weyl reflection $j \rightarrow \sigma^+(j) = -1 - j$. Besides, the quantity $\Delta_j - j$ is invariant under the *second* Weyl reflection $j \rightarrow \sigma^-(j) = 1 - k - j$.

The $sl(\hat{2})_k$ algebra (2.4) remains invariant under the following transformations

$$J_n^\pm \rightarrow \tilde{J}_n^\pm = J_{n \pm \omega}^\pm, \quad J_0^3 \rightarrow \tilde{J}_0^3 = J_0^3 + \omega \frac{k}{2}$$

for $\omega \in \mathbb{Z}$. This automorphism is called *spectral flow* and generates an infinite set of new representations, which we denote as $\mathcal{D}_j^{\pm, \omega}$, $\mathcal{C}_t^{\alpha, \omega}$ and $\mathcal{E}_j^{\alpha, \omega}$ with the intention to explicitly refer to the sector ω containing them as proper primary states. More precisely, the states of these *new* representations can be labelled by an additional index ω and are given by vectors which are Kac-Moody primaries with respect to the *new* generators (*i.e.* the \tilde{J}_n^a). These states are also primary states in the original Virasoro algebra, satisfying

$$L_0 |j, m, \omega\rangle = \Delta_j^\omega |j, m, \omega\rangle, \quad L_{n>0} |j, m\rangle = 0 \quad (2.6)$$

where $\Delta_j^\omega = \Delta_j - m\omega - \frac{k}{4}\omega^2$. Then, the Hilbert space is parametrized by a set of three quantum numbers, namely $|j, m, \omega\rangle$.

It is worth remarking that certain states belonging to the representations $\mathcal{D}_j^{\pm, \omega=0}$ are identified with states of the representation $\mathcal{D}_{-\frac{k}{2}-j}^{\mp, \omega=\pm 1}$, since the states of the form $|j, \mp j, 0\rangle$ coincide with $|\frac{k}{2}-j, \mp \frac{k}{2} \mp j, \pm 1\rangle$. Let us finally introduce $\sigma^0(j) = -\frac{k}{2} - j$, satisfying

$$\frac{1}{2}[\sigma^-, \sigma^+](j) = \pm[\sigma^\pm, \sigma^0](j) = k - 2. \quad (2.7)$$

Observe that the three transformations σ^a coincide in the *tensionless* limit $k \rightarrow 2$.

2.2 Vertex operator algebra

The vertex operators $\Phi_{j,m}(z)$ create the states $|j, m\rangle$ from the $SL(2, \mathbb{R})$ invariant vacuum $|0\rangle$; these are defined by the action

$$\lim_{z \rightarrow 0} \Phi_{j,m}(z) |0\rangle = |j, m\rangle \quad (2.8)$$

These are local operators which can be associated to differentiable functions $\Phi_j(z|x)$ on the manifold. We have that

$$[J_n^a, \Phi_j(z|x)] = z^n D_j^a \Phi_j(z|x) \quad (2.9)$$

where $a = \{+, -, 3\}$ and

$$D^3 = x\partial_x - j, \quad D^- = -\partial_x, \quad D^+ = -x^2\partial_x + 2jx \quad (2.10)$$

form a representation of $sl(2, \mathbb{R})$.

Then, the eigenfunctions $\psi_{j,m}^a(x)$ of the operators D^a are given by

$$\psi_{j,m}^3(x) = x^{j+m}, \quad \psi_{j,m}^\pm(x) = x^{j \mp j} e^{\pm mx^{\pm 1}}.$$

Let us consider $\psi_{j,m}^3(x)$. It is easy to show that this basis corresponds to the $sl(\hat{2})_k$ block structure for the states $|j, m\rangle$ presented in (2.5). In particular, the zero modes satisfy the following product

$$[J_0^3, \Phi_{j,m}] = m\Phi_{j,m}, \quad [J_0^\pm, \Phi_{j,m}] = (\mp j - m)\Phi_{j, m \pm 1}$$

which mimics the fact that

$$D^3\psi_{j,m}^3 = m\psi_{j,m}^3, \quad D^\pm\psi_{j,m}^3 = (\mp j - m)\psi_{j,m\pm 1}^3, \quad D^\pm\psi_{j,m}^\pm = m\psi_{j,m}^\pm. \quad (2.11)$$

Then, the representations $\Phi_j(z|x)$ are given by meromorphic functions in x and can be considered as the Fourier transform of the representations $\Phi_{j,m}(z)$, which, by using the basis of eigenfunctions $\psi_{j,m}^3(x)$, can be written as the following spectral decomposition

$$\Phi_{j,m,\bar{m}}(z, \bar{z}) = \int d^2x x^{j+m} \bar{x}^{j+\bar{m}} \Phi_j(x, \bar{x}|z, \bar{z}) \quad (2.12)$$

where we have explicitly considered the antiholomorphic part of the $sl(\hat{2})_k \otimes \bar{sl}(\hat{2})_k$ algebra. In the context of the AdS_3/CFT_2 correspondence, $\Phi_j(z|x)$ corresponds to a bulk-boundary propagator [12], where (x, \bar{x}) are the coordinates of the space where the dual BCFT is formulated.

The $\Phi_j(x|z)$ are associated to differentiable functions on the group manifold. In the study of the $SL(2, \mathbb{R})_k$ WZNW model, since one has no direct access to them as in the case of models corresponding to euclidean target spaces, it is usual to investigate the properties of the observables by considering the analytic continuation of the model on $SL(2, \mathbb{C})/SU(2)$. An example of this is the study of the two and three-point functions in string theory on AdS_3 [2], where the states of the model on $SL(2, \mathbb{R})$ appear as pole conditions of the analytic extension of the results obtained for the euclidean model $SL(2, \mathbb{C})/SU(2)$. Likewise, vertex operators are constructed by analytic continuation of the wave functions in the homogeneous space $SL(2, \mathbb{C})/SU(2)$.

A convenient representation for these wave functions can be given in terms of the Gauss parametrization of the group elements, namely

$$\Phi_j(x|z) = \frac{2j+1}{\pi} (|\gamma - x|^2 e^\phi + e^{-\phi})^{2j} \quad (2.13)$$

where $\gamma \in \mathbb{C}$ and $\phi \in \mathbb{R}$. In the quantum case, ϕ receives corrections as $\phi \rightarrow \phi/\sqrt{k-2}$.

Next, we will introduce the reducible representations of $SL(2, \mathbb{R})_k$, which are the central element of the discussion.

2.3 Degenerate representations

Kac and Kazhdan [13] found that a highest weight representation of $sl(2)_k$ is reducible if the highest weight j takes the values

$$j_{m,n}^+ = \frac{m-1}{2} + \frac{n-1}{2}(k-2), \quad j_{m,n}^- = -\frac{m+1}{2} - \frac{n}{2}(k-2). \quad (2.14)$$

In particular, there exists a null vector $|\chi_{m,n}^\pm\rangle$ with dimension

$$\Delta_{m,n}^+ = \Delta_{m,n} + m(n-1), \quad \Delta_{m,n}^- = \Delta_{m,n} + mn,$$

and charge

$$\tilde{j}_{m,n}^+ = j_{m,n}^+ - m = -\frac{m+1}{2} + \frac{n-1}{2}(k-2), \quad \tilde{j}_{m,n}^- = j_{m,n}^- + m = \frac{m-1}{2} - \frac{n}{2}(k-2), \quad (2.15)$$

respectively. More precisely, the null states that are present in the Verma module ² of these representations are given by [13, 14]

$$|\chi_{n,m}^\pm\rangle = \bar{K}_{m,n}^\pm K_{m,n}^\pm |j_{m,n}^\pm\rangle \quad (2.16)$$

where the *decoupling* operators $K_{m,n}^\pm$ can be written as follows

$$K_{m,n}^+ = (J_0^-)^{m-(n-1)(k-2)} (J_{-1}^+)^{m-(n-2)(k-2)} \dots (J_{-1}^+)^{m+(n-2)(k-2)} (J_0^-)^{m+(n-1)(k-2)} \quad (2.17)$$

$$K_{m,n}^- = (J_{-1}^+)^{m-(n-1)(k-2)} (J_0^-)^{m-(n-2)(k-2)} \dots (J_0^-)^{m+(n-2)(k-2)} (J_{-1}^+)^{m+(n-1)(k-2)} \quad (2.18)$$

By (2.16)-(2.18) and (2.9,2.10), the decoupling conditions for null states translate into differential equations to be satisfied by correlation functions involving $\Phi_{j_{m,n}^\pm}$.

As we have already mentioned in the introduction, we will refer to the states labelled by $j_{m,1}^+$ as the *classical branch*, because they have a classical limit. In this branch, the *decoupling* differential equations simply reflect the fact that the wave functions corresponding to $j_{m,1}^\pm$ are polynomials in the (x, \bar{x}) coordinates. In fact ³

$$\bar{K}_{m,1}^+ K_{m,1}^+ |j_{m,1}^+\rangle = 0 \quad \rightarrow \quad \partial_{\bar{x}}^m \partial_x^m \Phi_{\frac{m-1}{2}}(x|z) = 0. \quad (2.19)$$

²degenerate representations.

³where the l.h.s. of (2.19) has to be understood schematically as representing decoupling conditions of null states.

Let us also notice the following properties holding for degenerate representations

$$\tilde{j}_{m,1}^\pm = \sigma^\pm(j_{m,1}^\pm) , \quad j_{m,n}^\pm = \sigma^0(j_{m,n}^\mp). \quad (2.20)$$

Reducible representations are the fundamental elements in our discussion. The other ingredient which turns out to be important in the analysis is the expression of three-point correlation function, which is given in the following subsection.

2.4 Three-point correlation functions

The expressions of two and three-point functions in the gauged $SL(2, \mathbb{C})/SU(2)$ WZNW model were computed in [8]. In [2], the interpretation of these correlators as those describing string scattering amplitudes in AdS_3 was carefully carried out. For a complete discussion of three-point functions in $SL(2, \mathbb{R})$ see [3, 2].

The three-point correlation function can be written as follows

$$\langle \Phi_{j_1}(x_1|z_1) \Phi_{j_2}(x_2|z_2) \Phi_{j_3}(x_3|z_3) \rangle = \mathcal{A}_{j_1, j_2, j_3}$$

where

$$\mathcal{A}_{j_1, j_2, j_3} = \prod_{r < s} |z_r - z_s|^{2(\sum_{t=1}^3 \Delta_{j_t} - 2\Delta_{j_r} - 2\Delta_{j_s})} |x_r - x_s|^{2(2j_r + 2j_s - \sum_{t=1}^3 j_t)} C(j_1, j_2, j_3) \quad (2.21)$$

with $r, s, t \in \{1, 2, 3\}$ and

$$C(j_1, j_2, j_3) = \frac{1}{2\pi^3 b^2} \left(\lambda^{-1} \pi \frac{\Gamma(1 - b^2)}{\Gamma(1 + b^2)} \right)^{2 + \sum_r j_r} \frac{G(1 + \sum_{s=1}^3 j_s)}{G(-1)} \prod_{r=1}^3 \frac{G(-2j_r + \sum_{s=1}^3 j_s)}{G(1 + 2j_r)}$$

In the expression above, $b^{-2} = (k - 2)$, while λ is the coupling constant of the screening charge in $SL(2, \mathbb{R})_k$ WZNW model, which basically corresponds to the string coupling g_s^{-2} [15]. The $G(x)$ functions are defined as follows

$$G(x) = b^{-b^2 x^2 - (1+b^2)x} \Upsilon^{-1}(-bx) \quad (2.22)$$

where

$$\log \Upsilon(x) = \frac{1}{4} \int_0^\infty \frac{d\tau}{\tau} (b + b^{-1} - 2x)^2 e^{-\tau} - \int_0^\infty \frac{d\tau}{\tau} \frac{\sinh^2\left(\frac{\tau}{4}(b + b^{-1} - 2x)\right)}{\sinh\left(\frac{b\tau}{2}\right) \sinh\left(\frac{b^{-1}\tau}{2}\right)}$$

The $\Upsilon(x)$ functions were introduced in reference [16] in the context of Liouville theory, and we will make use of them below. These special functions have zeroes in the lattice

$$x \in -b\mathbb{Z}_{\geq 0} - b^{-1}\mathbb{Z}_{\geq 0} , \quad x \in b\mathbb{Z}_{>0} + b^{-1}\mathbb{Z}_{>0} \quad (2.23)$$

and satisfy the remarkable functional relation

$$\Upsilon(x + b^{\pm 1}) = \gamma(b^{\pm 1}x)b^{\pm 1 \mp 2b^{\pm}x}\Upsilon(x) \quad (2.24)$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

3 Logarithmic fields and the operator-valued relations

To begin with, let us introduce the special operators $\Phi'_j(x|z)$ defined as

$$\Phi'_j(x|z) = \frac{1}{2} \frac{\partial}{\partial j} \Phi_j(x|z) .$$

It is easy to observe that these operators together with Φ_j form a Jordan block with respect to the Virasoro algebra, which is encoded in the following operator product expansions [17]

$$T(z)\Phi_j(x|w) = \frac{\Delta_j}{(z-w)^2}\Phi_j(x|w) + \frac{1}{(z-w)}\partial_w\Phi_j(x|w) + \dots$$

$$T(z)\Phi'_j(x|w) = \frac{\Delta_j}{(z-w)^2}\Phi'_j(x|w) - \frac{\delta_j}{(z-w)^2}\Phi_j(x|w) + \frac{1}{(z-w)}\partial_w\Phi'_j(x|w) + \dots$$

where $2\delta_j = \frac{2j+1}{k-2}$. These operators Φ'_j are logarithmic fields in the CFT.

Consider the following discrete set of logarithmic fields associated to the degenerate Kac-Moody highest weight operators $\Phi_{j_{m,n}}$

$$\Phi'_{j_{m,n}}(x|z) = \frac{1}{2} \frac{\partial}{\partial j} \Phi_j(x|z)|_{j=j_{m,n}} .$$

These are the central elements in our discussion. The first crucial fact is that

$$\bar{K}_{m,n}^{\pm} K_{m,n}^{\pm} \Phi'_{j_{m,n}^{\pm}}(x, \bar{x}|z, \bar{z})$$

is a Kac-Moody highest weight operator.

The proof of this statement is similar to the one presented in [1] in Liouville field theory. Let us consider $\bar{K}_{m,n}^+ \Phi_j$. In a neighbourhood of $j = j_{m,n}^+$, $\bar{K}_{m,n}^+ \Phi_j = (j - j_{m,n}^+) A_{m,n} + \mathcal{O}((j - j_{m,n}^+)^2)$, where $A_{m,n}$ is an operator of dimension $(\Delta_{m,n} + m(n-1), \Delta_{m,n})$ and charge $j_{m,n}^+ - m$ under \bar{J}_0^3 [13, 14]. $A_{m,n}$ is not a left Kac-Moody primary operator any more but it is still a right Kac-Moody highest weight operator. It follows that $K_{m,n}^+ A_{m,n} = K_{m,n}^+ \bar{K}_{m,n}^+ \Phi'_{j_{m,n}^+}$ is also a right Kac-Moody highest weight operator. This is because $A_{m,n}$ has the same charge under J_0^3 and right dimension as $\Phi_{j_{m,n}^+}$. Inverting the roles of $K_{m,n}^+$ and $\bar{K}_{m,n}^+$, one can conclude that $K_{m,n}^+ \bar{K}_{m,n}^+ \Phi'_{j_{m,n}^+}$ is also a left Kac-Moody highest weight operator, of dimension $(\Delta_{m,n} + m(n-1), \Delta_{m,n} + m(n-1))$ and charge $(j_{m,n}^+ - m, j_{m,n}^+ - m)$ under \bar{J}_0^3, J_0^3 .

The second fact is that these are precisely the dimension and charge of $\Phi_{\tilde{j}_{m,n}^+}$, where

$$\tilde{j}_{m,n}^+ = j_{m,n}^+ - m = -\frac{m+1}{2} + \frac{n-1}{2}(k-2).$$

The same argument can be repeated for $j_{m,n}^-$. In this case, the dimension and charge of $K_{m,n}^- K_{m,n}^- \Phi'_{j_{m,n}^-}(x, \bar{x}|z, \bar{z})$ match those of $\Phi_{\tilde{j}_{m,n}^-}$.

In the following section, we will establish the operator-valued relation

$$\bar{K}_{m,n}^\pm K_{m,n}^\pm \Phi'_{j_{m,n}^\pm}(x, \bar{x}|z, \bar{z}) = B_{m,n}^\pm \Phi_{\tilde{j}_{m,n}^\pm}(x, \bar{x}|z, \bar{z}), \quad (3.25)$$

and evaluate the *Zamolodchikov coefficients* $B_{m,n}^\pm$.

We should mention that in the case of rational level $k-2 = p/q$ there exist two different null states in the module of $j_{m,n}^\pm$. This is due to the fact that $j_{m,n}^\pm = j_{m+p,n-q}^\pm$. In this particular case, we find two different operator-valued equations, one of these involving the spin $\tilde{j}_{m+p,n-q}^\pm = \tilde{j}_{m,n}^\pm - m$ primary field.

Before we proceed, let us show that the above identity holds in the classical limit for the operators $\Phi_{j_{m,1}^+}$ belonging to the *classical branch*. This amounts to showing that $\partial_x^m \partial_{\bar{x}}^m \Phi'_{j_{m,1}^+}$ is proportional to $\Phi_{\tilde{j}_{m,1}^+}$, where, by (2.13)

$$\Phi_{j_{m,1}^+} = \frac{m}{\pi} (|\gamma - x|^2 e^\phi + e^{-\phi})^{m-1}, \quad \Phi_{\tilde{j}_{m,1}^+} = -\frac{m}{\pi} (|\gamma - x|^2 e^\phi + e^{-\phi})^{-(m+1)}. \quad (3.26)$$

Note that

$$\partial_x^m \partial_{\bar{x}}^m \Phi'_{j_{m,1}^+} = \partial_x^m \partial_{\bar{x}}^m \left(\frac{m}{\pi} e^{-(m-1)\phi} A^{m-1} \log A \right), \quad (3.27)$$

where

$$A(x, \bar{x}) = |\gamma - x|^2 e^{2\phi} + 1. \quad (3.28)$$

Then, observing that

$$\partial_x^r (A^{r-1} \log A) = (r-1)! A^{-1} (\partial_x A)^r, \quad (3.29)$$

and using the identities

$$\partial_{\bar{x}}^s A^{-1} = (-1)^s s! A^{-s-1} (\partial_{\bar{x}} A)^s, \quad \partial_{\bar{x}}^s (\partial_x A)^r = \frac{r!}{(r-s)!} (\partial_x A)^{r-s} (\partial_x \partial_{\bar{x}} A)^s, \quad (3.30)$$

one finds

$$\partial_{\bar{x}}^r \partial_x^r (A^{r-1} \log A) = r! (r-1)! A^{-r-1} e^{2r\phi}, \quad (3.31)$$

which implies

$$\partial_{\bar{x}}^m \partial_x^m \Phi'_{j_{m,1}^+}(x, \bar{x}|z, \bar{z}) = -m! (m-1)! \Phi_{j_{m,1}^+}(x, \bar{x}|z, \bar{z}). \quad (3.32)$$

In the next section, we will evaluate the general expression of the *Zamolodchikov coefficients* $B_{m,n}^\pm$ and we will show that $B_{m,1}^+ \rightarrow -m!(m-1)!$ in the classical limit $k \rightarrow \infty$.

4 Zamolodchikov coefficients and fusion rules

Eq.(3.25) is an operator-valued relation, namely for every correlation function

$$\langle \bar{K}_{m,n}^+ K_{m,n}^+ \Phi'_{j_{m,n}^+}(x|z) \prod_{i=1}^{N-1} \Phi_{j_i}(x_i|z_i) \rangle = B_{m,n}^+ \langle \Phi_{j_{m,n}^+}(x|z) \prod_{i=1}^{N-1} \Phi_{j_i}(x_i|z_i) \rangle.$$

Thanks to the conformal invariance of the theory, it is sufficient to verify that the above equality holds for three-point functions. Therefore, we will compute the quotient between

$$\mathcal{A}'_{m,n} = \langle \bar{K}_{m,n}^+ K_{m,n}^+ \Phi'_{j_{m,n}^+}(x|z) \Phi_{j_1}(x_1|z_1) \Phi_{j_2}(x_2|z_2) \rangle$$

and

$$\tilde{\mathcal{A}}_{m,n} = \langle \Phi_{j_{m,n}^+}(x|z) \Phi_{j_1}(x_1|z_1) \Phi_{j_2}(x_2|z_2) \rangle,$$

which will yield the explicit form of coefficient $B_{m,n}^+$.

The latter correlator is simply given by

$$\begin{aligned} \tilde{\mathcal{A}}_{m,n} = C(\tilde{j}_{m,n}^+, j_1, j_2) & \frac{1}{|z - z_1|^{2(\tilde{\Delta}_{m,n} + \Delta_1 - \Delta_2)} |z_{12}|^{2(\Delta_1 + \Delta_2 - \tilde{\Delta}_{m,n})} |z - z_2|^{2(\Delta_2 + \tilde{\Delta}_{m,n} - \Delta_1)}} \\ & \times \frac{1}{|x - x_1|^{2(\tilde{j}_{m,n}^+ + j_1 - j_2 + 1)} |x_{12}|^{2(j_1 + j_2 - \tilde{j}_{m,n}^+ + 1)} |x - x_3|^{2(j_2 + \tilde{j}_{m,n}^+ - j_1 + 1)}}. \end{aligned}$$

Now, let us turn our attention to the three-point function $\mathcal{A}'_{m,n}$. First of all, note that $C(j, j_1, j_2)$ has a first order zero as $j \rightarrow j_{m,n}^+$. This implies that

$$\begin{aligned} & \langle \Phi'_{j_{m,n}^+}(x|z) \Phi_{j_1}(x_1|z_1) \Phi_{j_2}(x_2|z_2) \rangle \\ &= \frac{1}{2} \frac{\partial C(j, j_1, j_2) / \partial j|_{j=j_{m,n}^+}}{|z - z_1|^{2(\Delta_{m,n} + \Delta_1 - \Delta_2)} |z_{12}|^{2(\Delta_1 + \Delta_2 - \Delta_{m,n})} |z - z_2|^{2(\Delta_2 + \Delta_{m,n} - \Delta_1)}} \\ & \times \frac{1}{|x - x_1|^{2(j_{m,n}^+ + j_1 - j_2 + 1)} |x_{12}|^{2(j_1 + j_2 - j_{m,n}^+ + 1)} |x - x_3|^{2(j_2 + j_{m,n}^+ - j_1 + 1)}}. \end{aligned}$$

The operators $K_{m,n}^+$ and $\bar{K}_{m,n}^+$ change both the z and x dependence of the correlation function. They act on the holomorphic and antiholomorphic factors separately. Since the action of $K_{m,n}^+$ on a right Kac-Moody highest weight operator of charge $j_{m,n}^+$ and conformal dimension $\Delta_{m,n}$ produces another right Kac-Moody highest weight operator of charge $\tilde{j}_{m,n}^+ = j_{m,n}^+ - m$ and conformal dimension $\tilde{\Delta}_{m,n} = \Delta_{m,n} + m(n-1)$, we have that

$$\begin{aligned} & K_{m,n}^+ \left[(z - z_1)^{-(\Delta_{m,n} + \Delta_1 - \Delta_2)} (z_{12})^{-(\Delta_1 + \Delta_2 - \Delta_{m,n})} (z - z_2)^{-(\Delta_2 + \Delta_{m,n} - \Delta_1)} \right. \\ & \quad \left. \times (x - x_1)^{-(j_{m,n}^+ + j_1 - j_2 + 1)} (x_{12})^{-(j_1 + j_2 - j_{m,n}^+ + 1)} (x - x_3)^{-(j_2 + j_{m,n}^+ - j_1 + 1)} \right] \\ &= P_{m,n}(j_{m,n}^+, j_1, j_2) \left[(z - z_1)^{-(\tilde{\Delta}_{m,n} + \Delta_1 - \Delta_2)} (z_{12})^{-(\Delta_1 + \Delta_2 - \tilde{\Delta}_{m,n})} (z - z_2)^{-(\Delta_2 + \tilde{\Delta}_{m,n} - \Delta_1)} \right. \\ & \quad \left. \times (x - x_1)^{-(\tilde{j}_{m,n}^+ + j_1 - j_2 + 1)} (x_{12})^{-(j_1 + j_2 - \tilde{j}_{m,n}^+ + 1)} (x - x_3)^{-(j_2 + \tilde{j}_{m,n}^+ - j_1 + 1)} \right], \end{aligned}$$

where the function $P_{m,n}(j_{m,n}^+, j_1, j_2)$ is given by [5]

$$\begin{aligned} P_{m,n}(j_{m,n}^+, j_1, j_2) &= \prod_{r=0}^{m-1} \prod_{s=0}^{n-1} (j_{m,n}^+ + j_1 - j_2 - r - s(k-2)) \\ & \times \prod_{r=1}^m \prod_{s=1}^{n-1} (-j_{m,n}^+ + j_1 + j_2 + r + s(k-2)), \end{aligned} \tag{4.33}$$

and the equation $P_{m,n}(j_{m,n}^+, j_1, j_2) = 0$ yields precisely the fusion rules for the degenerate Kac-Moody primary $\Phi_{m,n}^+$ [5]. Repeating this argument for $\bar{K}_{m,n}^+$, we find that

$$\mathcal{A}'_{m,n} = \frac{P_{m,n}^2(j_{m,n}^+, j_1, j_2) \frac{1}{2} \partial C(j, j_1, j_2) / \partial j|_{j=j_{m,n}^+}}{|z - z_1|^{2(\tilde{\Delta}_{m,n} + \Delta_1 - \Delta_2)} |z_{12}|^{2(\Delta_1 + \Delta_2 - \tilde{\Delta}_{m,n})} |z - z_2|^{2(\Delta_2 + \tilde{\Delta}_{m,n} - \Delta_1)}} \\ \times \frac{1}{|x - x_1|^{2(\tilde{j}_{m,n}^+ + j_1 - j_2 + 1)} |x_{12}|^{2(j_1 + j_2 - \tilde{j}_{m,n}^+ + 1)} |x - x_3|^{2(j_2 + \tilde{j}_{m,n}^+ - j_1 + 1)}} .$$

Then we write the quotient of both correlators as follows

$$\frac{\mathcal{A}'_{m,n}}{\tilde{\mathcal{A}}_{m,n} P_{m,n}^2(j_{m,n}^+, j_1, j_2)} = \frac{1}{2} \frac{\partial C(j, j_1, j_2) / \partial j|_{j=j_{m,n}^+}}{C(\tilde{j}_{m,n}^+, j_1, j_2)} = -b \left(\frac{\lambda \gamma(b^2)}{\pi} b^{2b^2} \right)^{(\tilde{j}_{m,n} - j_{m,n})} \frac{\Upsilon'(-2bj_{m,n})}{\Upsilon(-2b\tilde{j}_{m,n})} \\ \times \frac{\gamma(-(\tilde{j}_{m,n} + j_1 + j_2 + k - 1)) \gamma(-b^2(2\tilde{j}_{m,n} + 1))}{\gamma(-(j_{m,n} + j_1 + j_2 + k - 1)) \gamma(-b^2(2j_{m,n} + 1))} \\ \times \frac{\Upsilon(-b(\tilde{j}_{m,n} + j_1 + j_2 + k - 1)) \Upsilon(-b(\tilde{j}_{m,n} + j_1 - j_2)) \Upsilon(-b(j_1 + j_2 - \tilde{j}_{m,n})) \Upsilon(-b(j_2 + \tilde{j}_{m,n} - j_1))}{\Upsilon(-b(j_{m,n} + j_1 + j_2 + k - 1)) \Upsilon(-b(j_{m,n} + j_1 - j_2)) \Upsilon(-b(j_1 + j_2 - j_{m,n})) \Upsilon(-b(j_2 + j_{m,n} - j_1))} , \quad (4.34)$$

where $Q = b + b^{-1}$ and $\Upsilon'(x) = \frac{d\Upsilon}{dx}(x)$. Here we wrote the expression in terms of the $\Upsilon(x)$ functions introduced before.

The last term in (4.34) is equal to

$$\frac{\Upsilon(-b(\tilde{j}_{m,n} + j_1 + j_2 + k - 1)) \Upsilon(-b(\tilde{j}_{m,n} - (j_1 + j_2 + k - 1)))}{\Upsilon(-b(j_{m,n} + j_1 + j_2 + k - 1)) \Upsilon(-b(j_{m,n} - (j_1 + j_2 + k - 1)))} \\ \times \frac{\Upsilon(-b(\tilde{j}_{m,n} + j_1 - j_2)) \Upsilon(-b(\tilde{j}_{m,n} + j_2 - j_1))}{\Upsilon(-b(j_{m,n} + j_1 - j_2)) \Upsilon(-b(j_{m,n} + j_2 - j_1))} , \quad (4.35)$$

where we used the identity $\Upsilon(Q - x) = \Upsilon(x)$.

By using (2.24), it can be also proven that

$$\frac{\Upsilon(-b(\tilde{j}_{m,n} + x)) \Upsilon(-b(\tilde{j}_{m,n} - x))}{\Upsilon(-b(j_{m,n} + x)) \Upsilon(-b(j_{m,n} - x))} = \frac{(-1)^{nm}}{p_{m,n}^2(x)} ,$$

where

$$p_{m,n}(x) = b^{mn} \prod_{r,s} \left(x - \frac{r}{2} - \frac{s}{2} (k - 2) \right) , \quad (4.36)$$

and $r = \{-m+1, -m+3, \dots, m-3, m-1\}$, $s = \{-n+1, -n+3, \dots, n-3, n-1\}$. Then, by (4.36), equation (4.35) becomes

$$\frac{1}{b^{4mn} \tilde{P}^2(j_{m,n}^+, j_1, j_2)},$$

where

$$\begin{aligned} \tilde{P}_{m,n}(j_{m,n}^+, j_1, j_2) &= \prod_{r=0}^{m-1} \prod_{s=0}^{n-1} (j_{m,n}^+ + j_1 - j_2 - r - s(k-2)) \\ &\times \prod_{r=1}^m \prod_{s=1}^n (-j_{m,n}^+ + j_1 + j_2 + r + s(k-2)). \end{aligned} \quad (4.37)$$

On the other hand, by using standard formulae involving $\Gamma(x)$ functions, we also find

$$\begin{aligned} R_{m,n} &\equiv \frac{\gamma(-(\tilde{j}_{m,n} + j_1 + j_2 + k - 1))}{\gamma(-(j_{m,n} + j_1 + j_2 + k - 1))} = \frac{\gamma(-(j_{m,n} + j_1 + j_2 + k - 1) + m)}{\gamma(-(j_{m,n} + j_1 + j_2 + k - 1))} \\ &= (-1)^m \left(\prod_{i=0}^{m-1} (j_{m,n} + j_1 + j_2 + k - 1 - i) \right)^2. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\mathcal{A}'_{m,n}}{\tilde{\mathcal{A}}_{m,n} P_{m,n}^2(j_{m,n}^+, j_1, j_2)} &= \frac{1}{2} \frac{\partial C(j, j_1, j_2) / \partial j|_{j=j_{m,n}^+}}{C(\tilde{j}_{m,n}^+, j_1, j_2)} = - \left(\frac{\lambda \gamma(b^2)}{\pi} b^{2b^2} \right)^{-m} \frac{\Upsilon'(-2bj_{m,n})}{\Upsilon(-2b\tilde{j}_{m,n})} \\ &\times \frac{\gamma(-b^2(2\tilde{j}_{m,n} + 1))}{\gamma(-b^2(2j_{m,n} + 1))} \frac{R_{m,n}}{b^{4mn-2} \tilde{P}_{m,n}^2(j_{m,n}^+, j_1, j_2)} \\ &= \left(\frac{\lambda \gamma(b^2)}{\pi} b^{2b^2} \right)^{-m} \frac{\Upsilon'(-2bj_{m,n})}{\Upsilon(-2b\tilde{j}_{m,n})} \frac{\gamma(-b^2(2\tilde{j}_{m,n} + 1))}{\gamma(-b^2(2j_{m,n} + 1))} \frac{b^{2-4mn} (-1)^{m+1}}{P_{m,n}^2(j_{m,n}^+, j_1, j_2)}, \end{aligned} \quad (4.38)$$

where we used

$$\frac{\tilde{P}_{m,n}^2}{P_{m,n}^2} = (-1)^m R_{m,n}.$$

Therefore, we find that any dependence of the ratio $\mathcal{A}'_{m,n} / \tilde{\mathcal{A}}_{m,n}$ on j_1, j_2 drops out and we obtain

$$B_{m,n}^+ = \frac{\mathcal{A}'_{m,n}}{\tilde{\mathcal{A}}_{m,n}} = \left(\frac{\lambda \gamma(b^2)}{\pi} b^{2b^2} \right)^{-m} \frac{\Upsilon'(-2bj_{m,n}^+)}{\Upsilon(-2b\tilde{j}_{m,n}^+)} \frac{\gamma(-b^2(2\tilde{j}_{m,n}^+ + 1))}{\gamma(-b^2(2j_{m,n}^+ + 1))} \times b^{2-4mn} (-1)^{m+1}. \quad (4.39)$$

The calculation for the case $j_{m,n}^-$ follows the same lines as the previous one. Then, we find

$$B_{m,n}^- = \left(\frac{\lambda \gamma(b^2)}{\pi} b^{2b^2} \right)^m \frac{\Upsilon'(-2bj_{m,n}^-)}{\Upsilon(-2bj_{m,n}^-)} \frac{\gamma(-b^2(2\tilde{j}_{m,n}^- + 1))}{\gamma(-b^2(2j_{m,n}^- + 1))} \times b^{2-4m(n-1)} (-1)^{m+1}. \quad (4.40)$$

In order to compare the above equations with the analogous results obtained in [1] for Liouville CFT, one can further simplify the above expressions and write

$$B_{m,n}^+ = \left(\frac{\lambda \gamma(b^2)}{\pi} \right)^{-m} (-1)^{m+1} b^{2-2n-4mn} \gamma(n + mb^2) \prod_{p=1-n}^{p=n-1} \prod_{q=1-m}^{q=m-1} (pb^{-1} + qb), \quad (4.41)$$

where $(p, q) \neq (0, 0)$.

Note that the classical limit, $b \rightarrow 0$, for the Zamolodchikov coefficient in *classical branch* is

$$\lim_{b \rightarrow 0} B_{m,1}^+ = -(\lambda^{-1} \pi)^m m! (m-1)! \quad (4.42)$$

which, upon fixing $\lambda = \pi$, exactly agrees with the classical result obtained in (3.32). Thus we see that the exact result (4.41) is fully consistent with the classical limit. Besides, even at finite b we have $B_{1,1}^+ = -\frac{\pi}{\lambda}$.

Finally, let us notice that Eq. (3.32), which is the classical limit of (3.25) in the case $(m, n) = (1, 1)$, can be rewritten as the Liouville equation $\partial_x \partial_{\bar{x}} \varphi(x|z) = e^{-2\varphi(x|z)}$ for the field $\varphi(x|z) = \log \Phi_{j_{2,1}}(x|z)$ in terms of the *boundary* variables (x, \bar{x}) .

5 Conclusions

In this letter, we derived an infinite set of operator-valued relations, Eq. (3.25), which hold for degenerate representations of $sl(\hat{2})_k$ Kac-Moody algebra. These relations are similar to those recently found by Zamolodchikov for Virasoro degenerate representations in Liouville conformal field theory [1].

By studying the functional form of the three-point functions on the sphere and considering the fusion rules of $sl(\hat{2})_k$ algebra we were able to find the explicit expression of the Zamolodchikov coefficients (4.39)-(4.40) for this non-rational CFT.

The operator-valued relations translate into differential equations satisfied by correlation functions involving particular Kac-Moody primary states. These are equations in terms

of the $SL(2, \mathbb{R})$ -isospin variables (x, \bar{x}) . Furthermore, the first equation of this infinite set resembles the Liouville equation. This observation could be relevant in the context of the AdS_3/CFT_2 correspondence since the variables (x, \bar{x}) precisely represent the coordinates of the boundary, where the dual conformal field theory is formulated. This could be an interesting topic for further research.

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